LIE GROUPS AND LIE ALGEBRAS BACKPAPER EXAMINATION

Total marks: 100

(1) Define a solvable, nilpotent and semisimple Lie algebra. Give an example (with proof) of each (the solvable Lie algebra should not be nilpotent). State the theorems of Engel and Lie. State Cartan's criteria for solvability and semisimplicity. (2 x 10 = 20 marks)

(2) Describe (without proof) the irreducible representations $V(m)$ of $\mathfrak{sl}(2, \mathbb{C})$ (for every non negative integer m). In particular, for each $V(m)$ write down a basis of weight vectors, and the explicit action of the elements h, x, y of $\mathfrak{sl}(2, \mathbb{C})$ on this basis. (10 marks)

(3) Describe a Cartan decomposition for the semisimple Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. Write down the set of roots for the decomposition. For any two non proportional roots α, β, calculate explicitly the α string through β, and compute the Cartan integers $\beta(h_\alpha)$. (10+10 =20 marks)

(4) Define a linear Lie group G and its associated Lie algebra $\mathfrak{g}$. Prove that $\mathfrak{g}$ is closed under the bracket operation. Let $f : G \to G'$ be a differentiable homomorphism between two linear groups G and G', and let $\mathfrak{g}$ and $\mathfrak{g}'$ be their respective Lie algebras. Prove that f induces a homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{g}'$, satisfying $f \circ exp = exp \circ \phi$. (10 + 20 = 30 marks)

(5) Let G be a connected linear group, let $\mathfrak{g}$ be its Lie algebra. Let H be a connected subgroup of G and let $\mathfrak{h}$ be its Lie algebra. Prove that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, if and only if, H is a normal subgroup of G. (10 marks)

(6) Describe the fibres of the exponential map for the Lie group $SO(3)$ (with proof) and also show that it is surjective. (10 marks)

Date: May 18, 2017.